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IMPROVED SUDAKOV-TYPE BOUNDS FOR OPTIMAL CONFIDENCE LIMITS ON T--ETC(U)

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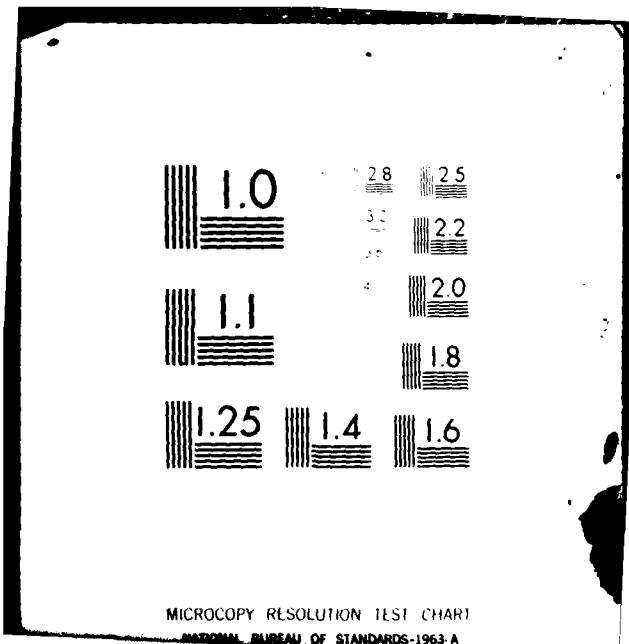
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IMPROVED SUDAKOV-TYPE BOUNDS FOR
OPTIMAL CONFIDENCE LIMITS ON THE
RELIABILITY OF SERIES SYSTEMS

Bernard Harris and Andrew P. Soms



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ABSTRACT

A sharper Sudakov-type lower bound for the lower confidence limit on the reliability of a series system than the one given in Harris and Soms (1980) is obtained. Numerical examples, coverage probabilities and the listings of the short FORTRAN programs used are also provided.

AMS (MOS) Subject Classifications: 62N05, 90B25.

Key Words: Optimal confidence bounds; Reliability; Series system.

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SIGNIFICANCE AND EXPLANATION

Serial Systems arise naturally in practice in engineering and physics. Therefore it is of substantial significance to be able to efficiently utilize data obtained on individual components for the purpose of obtaining an overall assessment of the reliability of the system. The methods of this paper may be employed for this purpose. As an example, suppose there are five subsystems, the sample sizes are (10, 15, 20, 25, 30) and the observed failures (1, 2, 1, 1, 2). Then the methods of this paper give that the optimum 95% lower confidence bound on the system reliability lies between .3871 and .3934.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

IMPROVED SUDAKOV-TYPE BOUNDS FOR
OPTIMAL CONFIDENCE LIMITS ON THE RELIABILITY
OF SERIES SYSTEMS

Bernard Harris* and Andrew P. Soms**

1. The Improved Sudakov-Type Bound

We will adhere to the notation of Harris and Soms (1980) and we refer the reader to this report for background material and additional references. Let V_i , $1 \leq i \leq k$, be independent binomial random variables with parameters n_i and p_i (the success probability) and let $X_i = n_i - V_i$, $q_i = 1 - p_i$, $1 \leq i \leq k$, with $n_1 \leq n_2 \leq \dots \leq n_k$ here and throughout. It is desired to obtain a "good" lower confidence limit on $\prod_{i=1}^k p_i$, the reliability of a series system with independent components. Let $g(\bar{x}) = g(x_1, x_2, \dots, x_k)$ be a monotonically decreasing or increasing (in each component) ordering function for real (not necessarily integral) x_i , $0 \leq x_i \leq n_i$, with large or small values, respectively, of $g(\bar{x})$ being best. Let $r_1 > r_2 > \dots > r_s$ be the ordered values of $g(\bar{x})$ in the decreasing case and $r_1 < r_2 < \dots < r_s$ in the increasing and let $A_i = \{\bar{x} | g(\bar{x}) = r_i\}$, $i = 1, 2, \dots, s$. Then (A_1, A_2, \dots, A_s) is a monotonic partition, i.e., $(0, 0, \dots, 0) \in A_1$, $(n_1, n_2, \dots, n_k) \in A_s$ and if $\bar{x}_1 = (x_{11}, \dots, x_{1k})$, $\bar{x}_2 = (x_{21}, \dots, x_{2k})$ with $x_{1i} \leq x_{2i}$, $i = 1, 2, \dots, k$, then $\bar{x}_1 \in A_i$ implies $\bar{x}_2 \in A_j$, $j \geq i$.

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Let

$$f(\tilde{x}; \tilde{p}) = P_{\tilde{p}}(\tilde{X} = \tilde{x}) = \prod_{i=1}^k \binom{n_i}{x_i} p_i^{n_i - x_i} q_i^{x_i} = \prod_{i=1}^k \binom{n_i}{v_i} p_i^{v_i} q_i^{n_i - v_i} \quad (2.1)$$

and for $1 \leq n \leq s-1$, let

$$a_n = \inf \left\{ \prod_{i=1}^k p_i \mid \sum_{i \leq n} \tilde{x}_j : \tilde{x}_j \in A_i \quad f(\tilde{x}_j; \tilde{p}) = \alpha \right\} \quad (2.2)$$

and $a_s = 0$. Each such partition may be identified with a function defined on the set of sample outcomes by defining the ordering function $g(\tilde{x})$, where

$$g(\tilde{x}) = n \quad \text{if} \quad \tilde{x} \in A_n, \quad 1 \leq n \leq s; \quad (2.3)$$

obviously $g(\tilde{x})$ inherits the monotonicity properties of the partition.

For defining Buehler's (1957) method of optimal lower confidence intervals on $\prod_{i=1}^k p_i$ we assume that $g(\tilde{x})$ has been redefined, if necessary, as in (2.3). Then we have, from Harris and Soms (1980),

Theorem 1. Let \tilde{X} be distributed by (2.1). Then $a_{g(\tilde{x})}$ is a $(1-\alpha)$ lower confidence bound for $\prod_{i=1}^k p_i$. If $b_{g(\tilde{x})}$ is also a $(1-\alpha)$ lower confidence bound for $\prod_{i=1}^k p_i$ which is monotonically decreasing in $g(\tilde{x})$, then $b_i \leq a_i$, $1 \leq i \leq s$.

We now let $g(\tilde{x})$ denote the original ordering function, since this is necessary for the applications below. In order to obtain bounds for $a_{g(\tilde{x}_0)}$ we must assume that \tilde{x}_0 is such that for each $t = 1, 2, \dots, k$, the equation

$$g(i_1, i_2, \dots, i_{t-1}, y_t, 0, \dots, 0) = g(\tilde{x}_0) \quad (2.4)$$

has a unique solution y_t , $y_t < n_t$, where i_r , $r = 1, 2, \dots, t-1$, are integers, $0 \leq i_r \leq y_r$. Define y_i^* by $g(0, 0, \dots, 0, y_i^*, 0, \dots, 0) = g(\tilde{x}_0)$.

where y_j^* is in the i^{th} position, the rest of the arguments being 0. Note that $y_1 = y_1^*$. A sufficient condition for (2.4) to hold is that

$g(\tilde{x})$ is strictly monotonic for $0 \leq x_j \leq y_j^* < n_j$, $j=1,2,\dots,k$

and

$$\lim_{x_r \rightarrow n_r} g(\tilde{x}) < g(\tilde{x}_0) \text{ for } g(\tilde{x}) \text{ decreasing} \quad \} \quad (2.5)$$

or

$$\lim_{x_r \rightarrow n_r} g(\tilde{x}) > g(\tilde{x}_0) \text{ for } g(\tilde{x}) \text{ increasing,} \quad r=1,2,\dots,k.$$

Assume now that (2.5) is satisfied and that in addition

$$\frac{y_{r-1}}{n_{r-1}} \geq \frac{y_r}{n_r}, \quad r=1,2,\dots,k-1, \quad (2.6)$$

where i_r integral, $0 \leq i_r \leq y_r$. Let

$$I_p(r,s) = \frac{1}{B(r,s)} \int_0^p t^{r-1} (1-t)^{s-1} dt,$$

and for $0 \leq y < n$, real, define $u(n,y,\alpha)$ by $\alpha = I_{u(n,y,\alpha)}(n-y, y+1)$.

Then it was shown in Harris and Soms (1980) that

$$u(n_1, y_1, \alpha) \leq a_{g(\tilde{x}_0)} \leq \min_{1 \leq i \leq k} u(n_i, [y_i^*], \alpha). \quad (2.7)$$

and thus if y_1 is an integer, $a_{g(\tilde{x}_0)} = u(n_1, y_1, \alpha)$.

Thus a conservative procedure is to use $u(n_1, y_1, \alpha)$ for $a_{g(\tilde{x}_0)}$. It is generally believed that this is quite conservative (see, e.g., Mann, Schafer and Singpurwalla, 1974). It is thus of practical interest to inquire whether the lower bound in (2.7) can be tightened. An examination of the proof of (2.7) in Harris and Soms (1980) shows this to be the case, with the new proof being coincident with the previous one, except for the omission of the final step. We have

Theorem 2. Under the same assumptions as for (2.7),

$$u'(n_1, n_2, y_1, y_2, \alpha) \leq a_g(\bar{x}_0) \leq \min_{1 \leq i \leq k} u(n_1, [y_i^*], \alpha),$$

where $u'(n_1, n_2, y_1, y_2, \alpha)$ is the solution in a of the equation

$$\sum_{i=1}^k \sup_{\sum_{j=1}^i p_j = a} \sum_{i_1=0}^{[y_1]} \binom{n_1}{n_1 - i_1} p_1^{n_1 - i_1} q_1^{i_1} \prod_{j=2}^k \binom{n_2 - y_2 + j - 1}{n_2 - y_2} p_j = \alpha.$$

It follows from the proof in Harris and Soms (1980) that

$$u(n_1, y_1, \alpha) \leq u'(n_1, n_2, y_1, y_2, \alpha).$$

Thus the only question is how great the improvement will be in using u' . The numerical examples in 2. show that this can be substantial.

Remarks. u' can be calculated quickly and efficiently by using a short FORTRAN program. The listing is given in the Appendix along with the listing of the program used to calculate coverage probabilities. Two ordering functions that satisfy (2.5) and (2.6) are $g(\bar{x}) = \sum_{i=1}^k ((n_i - x_i)/n_i)$ if $g(\bar{x}_0) > 0$ and $g(\bar{x}) = \sum_{i=1}^k x_i/n_i$ for sufficiently large n_i , $1 \leq i \leq k$ (see Harris and Soms, 1980, for details).

2. Coverage Probabilities and Numerical Examples

The ordering function used here throughout is

$g(\bar{x}) = \sum_{i=1}^k ((n_i - x_i)/n_i)$. Table 1 gives the coverage probabilities for $k = 3$, $\bar{n} = (5, 7, 10)$, $\alpha = .10$ and selected $\bar{p} = (p_1, p_2, p_3)$, for both the lower LB and upper UB bounds of (2.7). While the optimality property of Theorem 1 implies that there are \bar{p} for which the coverage probability is less than .9, it does not seem that there

are very many such \tilde{p} .

1. Coverage Probabilities for $\tilde{n} = (5, 7, 10)$ and $\alpha = .10$

<u>\tilde{p}</u>	<u>Coverages</u>	
	<u>UB</u>	<u>LB</u>
(.95,.95,.95)	.9999	.9999
(.95,.95,.90)	1.0000	1.0000
(.95,.95,.85)	1.0000	1.0000
(.95,.95,.80)	.9999	.9999
(.95,.95,.75)	.9999	.9999
(.95,.95,.70)	1.0000	1.0000
(.95,.95,.65)	.9535	.9927
(.95,.95,.60)	.9737	.9967
(.95,.95,.55)	.9869	.9874
(.95,.95,.50)	.9940	.9940

Table 2 gives some comparisons of LB, UB, the improved lower bound LBI and the true value TV for $k = 2$, $\alpha = .05$ and selected $\tilde{n} = (n_1, n_2)$ and $\tilde{x}_0 = (x_{01}, x_{02})$. LBI gives substantial improvement when n_1 is small compared to n_2 and little or none if n_1 is approximately the same as n_2 . The TV for $\tilde{n} = (5, 5)$ and $\tilde{x}_0 = (1, 1)$ agrees with the value in Lipow and Riley (1959).

2. Comparison of Bounds and True Value for $k = 2$ and $\alpha = .05$

<u>\tilde{n}</u>	<u>\tilde{x}_0</u>	<u>LB</u>	<u>LBI</u>	<u>TV</u>	<u>UB</u>
(5,5)	(1,1)	.2166	.2166	.2776	.3426
(5,10)	(1,1)	.2761	.3317	.3317	.3426
(5,10)	(2,3)	.0859	.1521	.1529	.1893
(10,20)	(1,2)	.5037	.5675	.5691	.6058
(10,20)	(4,5)	.1851	.2137	.2137	.2224

Table 3 gives LB, UB and LBI for $k = 5$, $\alpha = .05$ and selected \tilde{n} and \tilde{x} .

3. Comparison of Bounds for $k = 5$ and $\alpha = .05$

<u>\tilde{n}</u>	<u>\tilde{x}_0</u>	<u>LB</u>	<u>LBI</u>	<u>UB</u>
(10,10,10,10,10)	(1,2,0,1,1)	.2893	.2893	.3035
(10,15,20,25,30)	(1,2,1,1,2)	.3599	.3871	.3934
(10,15,20,25,30)	(1,1,2,3,4)	.2838	.3017	.3035
(10,15,20,25,30)	(2,3,4,4,4)	.1319	.1478	.1500

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Lipow, M. and Riley, J. (1959), "Tables of Upper Confidence Bounds on Failure Probability of 1, 2, and 3 Component Serial Systems", Vols. I and II, Space Technology Laboratories.

Mann, Nancy R., Schafer, Ray E., and Singpurwalla, Nozer D. (1974), Methods for Statistical Analysis of Reliability and Life Data, New York: John Wiley and Sons.

Appendix

The listings of the coverage and improved bound FORTRAN programs are given below.

```
N'S SAMPLE SIZES FROM SMALLEST TO LARGEST, P'S SUCCESS PROBABILITIES
K NUMBER OF SUBSYSTEMS, 1-ALPHA CONFIDENCE LEVEL
DIMENSION P(10),N(20),NY(10), PLIMU(10),PI(10,41),LL(10),II(10)
100 FORMAT ( )
EQUIVALENCE (I1,II(1)),(I2,II(2)),(I3,II(3)),(I4,II(4)),
1(I5,II(5)),(I6,II(6)),(I7,II(7)),(I8,II(8)),(I9,II(9)),(I10,II(10)
2)
1 READ 100,K,ALPHA
  IF (K.EQ.0) GO TO 21
  DO 221 I=1,10
    LL(I)=0
221 N(I)=0
  READ 100,(N(I),P(I),I=1,K)
  DO 343 II1=1,K
    DO 343 II2=0,N(II1)
      CALL MDBIN(II2,N(II1),P(II1),PT,PI(II1,II2+1),IER)
343 IF (PT.LE..0001) LL(II1)=LL(II1)+1
  PARAM=1.
  IPROD2=1
  DO 11 I=1,K
    IPROD2=IPROD2*N(I)
11 PARAM=PARAM*P(I)
  PRL=0.
  PRU=0.
  DO 122 I1=LL(1),N(1)
    DO 122 I2=LL(2),N(2)
    DO 122 I3=LL(3),N(3)
    DO 122 I4=LL(4),N(4)
    DO 122 I5=LL(5),N(5)
    DO 122 I6=LL(6),N(6)
```

```

DO 122 I7=LL(7),N(7)
DO 122 I8=LL(8),N(8)
DO 122 I9=LL(9),N(9)
DO 122 I10=LL(10),N(10)
PROBT =1.
DO 511 IT=1,K
INT=II(IT)
511 PROBT=PROBT*PI(IT,INT+1)
IPROD1=1
DO 4 IJ=1,K
IN=II(IJ)
4 IPROD1=IPROD1*IN
IF (IPROD1.EQ.0) GO TO 15
ITEM1=IPROD2-IPROD1
Y1=N(1)*ITEM1/FLOAT(IPROD2)
DO 5 IT=1,K
NY(IT)=N(IT)*ITEM1/IPROD2
A=FLOAT(N(IT)-NY(IT))
R=FLOAT(NY(IT)+1)
5 CALL MDBETA(PARAM,A,B,PLIMU(IT),IER)
CALL MDBETA(PARAM,N(1)-Y1,Y1+1.,PLIML,IER)
IF (PLIML.GE.ALPHA) PRL=PRL+PROBT
IFLAG=0
DO 6 IT=1,K
6 IF (PLIMU(IT).GE.ALPHA) IFLAG=1
IF (IFLAG.EQ.1.) PRU=PRU+PROBT
GO TO 122
15 PRL=PRL+PROBT
PRU=PRU+PROBT
122 CONTINUE
PRINT 100,K,ALPHA,(N(I),P(I),I=1,K)
PRINT 100,PRU,PRL
GO TO 1
21 STOP
END

```

```

C SERIES WITH UNLIKE COMPONENTS
C MDBIN, MDBETA, MDBETI IMSL BINOMIAL, BETA, INVERSE BETA ROUTINES,
C N'S SAMPLE SIZES FROM SMALLEST TO LARGEST, X'S FAILURES, NINT NUMBER
C OF P VALUES CONSIDERED FOR SUP, EPS MAXIMUM ERROR IN SOLUTION FOR
C REFINED BOUND, NO X'S EQUAL TO N'S

    REAL ITEM1,ITEM2,ITEM3,ITEMT
    DIMENSION N(50),X(50)

100 FORMAT ( )
    DATA EPS/.001/
    I READ 100,K,ALPHA,NINT
    IF (K.EQ.0) GO TO 99
    READ 100,(N(I),X(I),I=1,K)
52 PRINT 100,'DATA',(N(I),X(I),I=1,K)
    PRINT 100,'ALPHA=',ALPHA
    ITEM1=1.
    ITEM2=1.
    DO 2 I=1,K
    ITEM1=ITEM1*N(I)
2 ITEM2=ITEM2*(N(I)-X(I))
    ITEM3=N(1)*(ITEM1-ITEM2)
    Y1=ITEM3/ITEM1
    NY1=Y1
    CALL MDBETI(ALPHA,N(1)-Y1,1.,A,IER)

C COMPUTATION OF UPPER BOUND
    BU=1.
    DO 3 I=1,K
    ITEMT=N(I)*(ITEM1-ITEM2)
    NYT=ITEMT/ITEM1
    CALL MDBETI(ALPHA,FLOAT(N(I)-NYT),NYT+1.,BUT,IER)
3 IF (BUT.LT.BU) BU=BUT
    PRINT 100,'L-M LOWER BOUND =',A
    PRINT 100,'UPPER BOUND =',BU

-C COMPUTATION OF IMPROVED LOWER BOUND
    TEML=A
    TEMU=RU
9 TEM=(TEML+TEMU)/2.
    DELTA=(1-TEM)/NINT

```

```
SUP=0.  
DO 42 I=1,NINT  
P1=TEM+(I-1)*DELTA  
P2=TEM/P1  
Q1=1-P1  
PROB=0.  
DO 33 K1=0,NY1  
Y2=N(2)-(ITEM2/ITEM1)*(N(1)*N(2)/(N(1)-K1))  
CALL MDBIN(K1,N(1),Q1,PS1,PK1,IER)  
CALL MDBETA(P2,N(2)-Y2,Y2+1.,PP2,IER)  
33 PROB=PROB+PK1*PP2  
IF (PROB.GT.SUP) SUP=PROB  
42 CONTINUE  
IF (SUP.LE.ALPHA) TEML=TEM  
IF (SUP.GT.ALPHA) TEMU=TEM  
IF (ABS(TEMU-TEML).LE.EPS) GO TO 10  
GO TO 9  
10 A=TEML  
PRINT 100,'IMPROVED LOWER BOUND =',A  
GO TO 1  
99 STOP  
END
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